A CLASSIFICATION OF SMOOTH EMBEDDINGS OF 4-MANIFOLDS IN 7-SPACE, I

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Abstract. We work in the smooth category. Let N be a closed connected n-manifold and assume that $m > n + 2$. Denote by $E^m(N)$ the set of embeddings $N \to \mathbb{R}^m$ up to isotopy. The group $E^m(S^n)$ acts on $E^m(N)$ by embedded connected summation of a manifold and a sphere. If $E^m(S^n)$ is non-zero (which often happens for $2m < 3n + 4$) then until recently no complete readily calculable description of $E^m(N)$ or of this action were known (as far as I know). Our main results are *examples of the triviality and the effectiveness of this action*, and *a complete readily calculable isotopy classification of embeddings into* $\mathbb{R}^7$ *for certain 4-manifolds* N. The proofs use new approach based on the Kreck modified surgery theory and the construction of a new invariant.

Corollary. *(a) There is a unique embedding* $f : \mathbb{C}P^2 \to \mathbb{R}^7$ *up to isoposition (i.e. for each two embeddings* $f, f' : \mathbb{C}P^2 \to \mathbb{R}^7$ *there is a diffeomorphism* $h : \mathbb{R}^7 \to \mathbb{R}^7$ *such that* $f' = h \circ f$*).*

(b) For each embedding $f : \mathbb{C}P^2 \to \mathbb{R}^7$ *and each non-trivial embedding* $g : S^4 \to \mathbb{R}^7$ *the embedding* $f\#g$ *is isotopic to* f*.*

1. Introduction

Knotting Problem for 4-manifolds.

This paper is on the classical Knotting Problem: *given an n-manifold N and a number m, describe isotopy classes of embeddings* $N \to \mathbb{R}^m$. For recent surveys see [RS99, Sk08, HCEC]. We work in the smooth category.

For $2m \geq 3n + 4$ there are some complete readily calculable classifications of isotopy classes [Sk08, §2, §3, HCEC].[1] If

$$2m < 3n + 4$$

and a closed manifold N different from disjoint union of homology spheres, then until recently no complete readily calculable descriptions of isotopy classes were known (as

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[1] By readily calculable classification I mean a classification in terms of homology of a manifold (and certain structure in homology like intersection, characteristic classes etc.). A readily calculable classification is also a reduction to calculation of stable homotopy groups of spheres (or to another standard algebraic problem involving only homology of our manifold, which problem is solved in particular cases, although could be unsolved generally). An important feature of a useful classification is accessibility of statement to general mathematical audience which is only familiar with basic notions of the area.

far as I know), in spite of the existence of interesting approaches of Browder-Wall and Goodwillie-Weiss [Wa70, GW99, CRS04].[2] For recent results see [Sk06] (a classification of embeddings $S^p \times S^q \to \mathbb{R}^m$), [Sk08'] (a classification of embeddings of 3-manifolds into $\mathbb{R}^6$), [CS08] (see comments on this closely related paper after Lemma 1.3), [CRS07, CRS08] (rational classification). For *piecewise linear* classification see [Sk07, Sk08, §2, §3].

In particular, a complete readily calculable classification of embeddings of a closed connected 4-manifold N into $\mathbb{R}^m$ was only known either for $m \ge 8$ or for $N = S^4$ and $m = 7$ by Wu, Haefliger, Hirsch and Bausum:

$$\#E^m(N) = 1 \quad \text{for} \quad m \ge 9,$$

$$E^8(N) = \begin{cases} H_1(N;\mathbb{Z}_2) & N \text{ orientable} \\ \mathbb{Z} \oplus \mathbb{Z}_2^{s-1} & N \text{ non-orientable and } H_1(N;\mathbb{Z}_2) \cong \mathbb{Z}_2^s \end{cases},$$

$$E^7(S^4) \cong \mathbb{Z}_{12}.$$

Here $E^m(N)$ is the set of smooth embeddings $N \to \mathbb{R}^m$ up to smooth isotopy; the equality sign between sets denotes the existence of a bijection; the isomorphism is a group isomorphism for certain geometrically defined group structures. See references in [Sk08, §2, §3, HCEC]; cf. [CS08].[3]

One of the main results of this paper (the Triviality Theorem 1.1(b)) is *a complete readily calculable classification of embeddings of certain 4-manifolds into* $\mathbb{R}^7$, cf. [CS08].

Main results.

Consider the 'connected sum' group structure on $E^m(S^n)$ defined in [Ha66]. By [Ha61, Ha66, Corollary 6.6, Sk08, §3], $E^m(S^n) = 0$ for $2m \ge 3n + 4$. However, $E^m(S^n) \ne 0$ for many m, n such that $2m < 3n + 4$,[4] e.g. $E^7(S^4) \cong \mathbb{Z}_{12}$.[5]

In this and the next subsection we assume that N is a closed n-manifold and $m \ge n+3$. The group $E^m(S^n)$ acts on the set $E^m(N)$ by connected summation of embeddings $g : S^n \to \mathbb{R}^m$ and $f : N \to \mathbb{R}^m$ whose images are contained in disjoint cubes.[6] Various authors have studied analogous connected sum action of the group of homotopy n-spheres on the set of n-manifolds topologically homeomorphic to given manifold [Le70].

The quotient of $E^m(N)$ modulo the above action of $E^m(S^n)$ is known in some cases.[7] Thus in these cases the Knotting Problem is reduced to the description of the above

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[2] The author is grateful to M. Weiss for indicating that the approach of [GW99] does give explicit results on higher homotopy groups of the space of embeddings $S^1 \to \mathbb{R}^n$.

[3] The known *existence* results for closed 4-manifolds N are as follows:
- N embeds into $\mathbb{R}^8$;
- if N is orientable, then N embeds into $\mathbb{R}^7$ [Hi65, Fu94, cf. BH70, Fu01];
- N embeds into $\mathbb{R}^7$ if and only if $\overline{W}_3(N) = 0$ [Fu94];
- an orientable N PL embeds into $\mathbb{R}^6$ if and only if $\overline{w}_2(N) = 0$ [Ma78, Corollary 10.11, CS79].
- an orientable N smoothly embeds into $\mathbb{R}^6$ if and only if $\overline{w}_2(N) = 0$ and $\sigma(N) = 0$ [Ma78, Corollary 10.11, CS79, Ru82].

[4] This differs from the Zeeman-Stallings Unknotting Theorem: *for* $m \ge n+3$ *any PL or TOP embedding* $S^n \to S^m$ *is PL or TOP isotopic to the standard embedding.*

[5] This follows from [Ha66, 4.11, cf. Ha86] and well-known Lemma 3.1.

[6] Since $m \ge n + 3$, the connected sum is well-defined, i.e. does not depend onthe choice of an arc between gS^n and fN. If N is not connected, we assume that a component of N is chosen and we consider embeddied connected summation with this chosen component.

[7] In those cases when this quotient coincides with the set of PL embeddings $N \to \mathbb{R}^m$ up to PL isotopy and when the latter set was known [Sk08, §2, Sk02, Sk07].

action of $E^m(S^n)$ on $E^m(N)$. Until recently *no results were known* on this action for $E^m(S^n) \neq 0$ and N not a disjoint union of spheres. For recent results see [Sk06, Sk08', CS08]; for *rational* description see [CRS07, CRS08]; for $m = n+2$ see [Vi73].

The main results of this paper are the following *examples of the triviality and the effectiveness of the above action for embedings of 4-manifolds into* $\mathbb{R}^7$.

We omit $\mathbb{Z}$-coefficients from the notation of (co)homology groups.

The Triviality Theorem 1.1. *Let N be a closed connected smooth 4-manifold such that $H_1(N) = 0$ and the signature $\sigma(N)$ of N is free of squares (i.e. is not divisible by the square of an integer $s \geq 2$).*

(a) For each embeddings $f : N \to \mathbb{R}^7$ and $g : S^4 \to \mathbb{R}^7$ the embedding $f\#g$ is isotopic to f (although g could be non-isotopic to the standard embedding).

(b) There is a 1–1 correspondence

$$BH : E^7(N) \to \{x \in H_2(N) \mid x \mod 2 = PDw_2(N),\ x \cap x = \sigma(N)\}.$$

Here $PDw_2(N)$ is Poincaré dual of the 2nd Sitefel-Whitney class, and $\cap$ is the intersection product, cf. Remark 2.3.

E. g. $N = \mathbb{C}P^2$ satisfies to the assumption of the Triviality Theorem 1.1, so $E^7(\mathbb{C}P^2)$ is in 1–1 correspondence with $\{+1, -1\} \subset \mathbb{Z} \cong H_2(\mathbb{C}P^2)$.[8]

Let $N_0 := \mathrm{Cl}(N - B^4)$, where B^4 is a closed 3-ball in N.

The Effectiveness Theorem 1.2. *Let N be a closed smooth 4-manifold such that ΣN retracts to ΣN_0. Let $f_0 : N \to \mathbb{R}^7$ be an embedding such that $f_0 N \subset \mathbb{R}^6$ (thus N is embeddable into $\mathbb{R}^6$). Then for each non-isotopic embeddings $g_1, g_2 : S^4 \to \mathbb{R}^7$ the embedding $f_0\#g_1$ is not isotopic to $f_0\#g_2$.*

We have that ΣN retracts to ΣN_0 if N is spin (that is, $w_2(N) = 0$) and simply-connected [Mi58], or if $N = S^1 \times S^3$, or if N is a connected sum of manifolds with this property. It would be interesting to know whether ΣN retracts to ΣN_0 for any spin 4-manifold N.

Remark. Let N be a closed connected orientable smooth 4-manifold. Then the following conditions are equivalent:

(1) N is embeddable into $\mathbb{R}^6$;

(2) $w_2(N) = 0$ and $\sigma(N) = 0$.

(3) the normal bundle of each embedding $f : N \to S^7$ is trivial;

For simply-connected N each of these conditions is equivalent to

(4) the intersection form of N is that of $\#_i(S^2 \times S^2)$;[9]

(5) N is homotopy equivalent to $\#_i(S^2 \times S^2)$;

(6) N is topologically homeomorphic to $\#_i(S^2 \times S^2)$.

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[8] The two isotopy classes of embeddings $\mathbb{C}P^2 \to \mathbb{R}^7$ are represented by the standard embedding and by its composition with the reflection of $\mathbb{R}^7$. One of constructions of the standard embedding $\mathbb{C}P^2 \to \mathbb{R}^7$ is as follows [BH70, p. 164, Sk08', §5]. It suffices to construct *an embedding $f_0 : (\mathbb{C}P^2 - B^4) \to S^6$ such that the boundary 3-sphere is the standard one* (because then f_0 could be extended to a smooth embedding $\mathbb{C}P^2 \to S^7$ using bell-like functions). Recall that $\mathbb{C}P^2 - B^4$ is the mapping cylinder of the Hopf map $h : S^3 \to S^2$. Recall that $S^6 = S^2 * S^3$. Define $f_0[(x,t)] := [(x, h(x), t)]$, where $x \in S^3$. In other words, the segment joining $x \in S^3$ and $h(x) \in S^2$ is mapped onto the arc in S^6 joining x to $h(x)$. See also [Ca39, KK95, Ma75].

[9] In the connected sum in (4), (5) and (6) the number of summands could be zero.

Proof. (1) is equivalent to (2) by [GS99, Theorem 9.1.21 and Remark 9.1.22, cf. CS79, Ru82]. (2) is equivalent to (3) by the Dold-Whitney Theorem [DW59], cf. [CS08, the Normal Bundle Lemma].

By the Whitehead and the Freedman Theorems, (4) is equivalent to (5) and (6).

Clearly, (4) implies (2). For simply-connected N (2) implies that the intersection form of N is indefinite and even, so (4) holds by [Ma80, Theorem 1.9.2]. $\square$

Ideas of proof.

The Effectiveness Theorem 1.2 is proved in §3 (cf. Theorem 3.5). The proof is based on the construction of a new *attaching invariant* for certain embeddings $N \to \mathbb{R}^m$, generalizing the Haefliger-Levine attaching invariant of embeddings $S^n \to \mathbb{R}^m$.

The proof of the Triviality Theorem 1.1 is much more non-trivial. The proof is based on the following idea, which is useful not only in these dimensions [Sk08'] and not only to describe the action of $E^m(S^n)$ on $E^m(N)$ [FKV87, FKV88].

Fix an orientation on N and an orientation on $\mathbb{R}^m$. Denote by

- C_f the closure of the complement in $S^m \supset \mathbb{R}^m$ to a tubular neighborhood of fN,
- $\nu = \nu_f : \partial C_f \to N$ the restriction of the normal bundle of f.

Lemma 1.3. *For a closed connected manifold N embeddings $f, f' : N \to \mathbb{R}^m$ are isotopic if and only if there is an orientation-preserving bundle isomorphism $\varphi : \partial C_f \to \partial C_{f'}$ which extends to an orientation-preserving diffeomorphism $C_f \to C_{f'}$.*

Proof. The 'only if' part is obvious, so let us prove the 'if' part. The isomorphism φ also extends to an orientation-preserving diffeomorphism $S^m - \operatorname{Int} C_f \to S^m - \operatorname{Int} C_{f'}$. Hence φ extends to an orientation-preserving diffeomorphism $\mathbb{R}^m \cong \mathbb{R}^m$. Since any orientation-preserving diffeomorphism of $\mathbb{R}^m$ is isotopic to the identity, it follows that f and f' are isotopic. $\square$

So results on the Diffeomorphism Problem can be applied to Knotting Problem. In this way there were obtained embedding theorems in terms of Poincaré embeddings [Wa70]. But 'these theorems reduce geometric problems to algebraic problems which are even harder to solve' [Wa70]. One of the main problems is that in general (i.e. not in simpler cases like that of the Effectiveness Theorem 1.2) it is hard to work with the homotopy type of C_f (which is sometimes unknown even when the classification of embeddings is known [Sk06]). The main idea of our proof is to apply the modification of surgery [Kr99] which allows to classify m-manifolds using their homotopy type just below dimension $m/2$.

The relation of this paper to a closely related paper [CS08] is as follows. Shortly, the proof for the simplest (but non-trivial) cases (like $\mathbb{C}P^2$ in $\mathbb{R}^7$) is presented in this paper, while more complicated proof for more general case is given in [CS08]. More precisely, the main result of [CS08] is a description of $E^7(N)$ for each closed connected 4-manifold N such that $H_1(N) = 0$. This result recovers the Triviality Theorem 1.1 completely and the Effectiveness Theorem 1.2 only for the case $H_1(N) = 0$. The proof in this paper is almost disjoint from the proof in [CS08] and is much shorter. The difference in applying the modification of surgery [Kr99] is that here we use $BO\langle 5\rangle \times \mathbb{C}P^\infty$-surgery while in [CS08] $BSpin \times \mathbb{C}P^\infty$-surgery is used; the attaching invariant is not used in [CS08].

In the first subsection of §3 we present well-known definition of the attaching invariant $a : E^7(S^4) \to \mathbb{Z}_{12}$. This subsection is formally not used later in §3, where we generalize this definition. In §4 we give a new proof of its injectivity based on [Kr99]. Otherwise §3 and §4 are independent on each other.

The Triviality Theorem 1.1(a) follows by the Boéchat-Haefliger Theorem 2.1(a) and the Complement Lemma 2.2(b) of the following subsection, together with the following result, which is the new and the most important part of the proof.

The Primitivity Theorem 1.4. *Let N be a closed connected smooth 4-manifold and $f : N \to \mathbb{R}^7$ an embedding such that $\pi_3(C_f) = 0$ (and hence $H_1(N) = 0$). Then for each embedding $g : S^4 \to \mathbb{R}^7$ the embedding $f\#g$ is isotopic to f.*

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2. Preliminaries

The Boéchat-Haefliger invariant.

From now on, unless otherwise stated, we assume that

N is a closed connected orientable 4-manifold and $f : N \to \mathbb{R}^7$ is an embedding.

The *homology Seifert surface* A_f for f is the image of the fundamental class $[N]$ under an inverse to 'homology Alexander duality', i.e. to the composition $H_5(C_f, \partial C_f) \to H_4(\partial C_f) \to H_4(N)$ of the boundary map and the normal bundle map. (This composition equals to the composition $H_4(N) \to H^2(C_f) \to H_5(C_f, \partial C_f)$ of the Alexander and Poincaré-Lefschetz duality isomorphisms [Sk08', the Alexander Duality Lemma].)

Define *the Boéchat-Haefliger invariant*

$$BH : E^7(N) \to H_2(N)$$

by setting $BH(f)$ to be the image of $A_f \cap A_f$ under 'homology Alexander duality', i.e. under the composition $H_3(C_f, \partial C_f) \to H_2(\partial C_f) \to H_2(N)$ of the boundary map and the normal bundle map. (A definition of $A_f \cap A_f$ is recalled in Remark 2.3.)

This new definition is equivalent to the original one [BH70] by the Section Lemma 2.5.

The Boéchat-Haefliger Theorem 2.1.

$$(a) \qquad \operatorname{im} BH \;=\; \{x \in H_2(N) \mid x \mod 2 = PDw_2(N),\ x \cap x = \sigma(N)\}.$$

(b) *If $H_1(N) = 0$, then two embeddings $N \to \mathbb{R}^7$ with the same BH-invariant differ by a connected sum with an embedding $S^4 \to \mathbb{R}^7$.*

E. g. $H_2(S^2 \times S^2) \cong \mathbb{Z} \oplus \mathbb{Z} \supset \{(2k, 2l) \mid kl = 0\} = \operatorname{im} BH$ for $N = S^2 \times S^2$.

Here part (a) follows by the Section Lemma 2.5.b and [BH70, Theorem 2.1], cf. [Fu94]. Part (b) follows by the Section Lemma 2.5.b, [BH70, Theorem 1.6] and smoothing theory [BH70, p. 156], cf. [Ha67, Ha].

The Triviality Theorem 1.1(b) follows from the Triviality Theorem 1.1(a) and the Boéchat-Haefliger Theorem 2.1.

The Complement Lemma 2.2. *If $H_1(N) = 0$, then*

(a) $C_f \simeq C_{BH(f)} := S^2 \cup_{BH(f)} (D^4_1 \sqcup \dots \sqcup D^4_{b_2(N)})$. Here we identify $H_2(N)$ and $\mathbb{Z}^{b_2(N)}$ by any isomorphism, so $BH(f)$ is identified with an ordered set of $b_2(N)$ integers, which set defines a homotopy class of maps $\partial(D^4_1 \sqcup \dots \sqcup D^4_{b_2(N)}) \to S^2$.[10]

(b) $\pi_3(C_f) \cong \mathbb{Z}/d\mathbb{Z}$, where $d = 0$ for $BH(f) = 0$ and

$$d := \max\{k \in \mathbb{Z} \mid \text{there is } y \in H_2(N) : \ BH(f) = ky\} \quad \text{for} \quad BH(f) \neq 0.$$

Proof. Part (b) follows by (a). Let us prove (a).

By general position C_f is simply-connected. Since $H_1(N) = 0$, by Alexander duality

$$H^2(C_f) \cong \mathbb{Z}, \quad H^4(C_f) \cong H_2(N) \quad \text{and} \quad H^i(C_f) = 0 \quad \text{for} \quad i \neq 2, 4.$$

Now the Complement Lemma 2.2.a is obtained by taking $p = 1$ in the following statement implied by 'homology decomposition of Eckman-Hilton' [EH59].

If Y is a finite simply-connected cell-complex such that $H^i(Y) = 0$ for $i \neq 0, 2, 4$, $H^2(Y) \cong \mathbb{Z}^p$, $H^4(Y) \cong \mathbb{Z}^q$ and A is the $(p \times q)$-matrix of the cup square $H^2(Y) \to H^4(Y)$ in some bases of $H^2(Y)$ and $H^4(Y)$, then $Y \simeq (S^2_1 \vee \dots \vee S^2_p) \cup_A (e^4_1 \sqcup \dots \sqcup e^4_q)$. $\square$[11]

Section Lemma 2.5.

In this subsection we omit index f from the notation.

Remark 2.3. In this paper we mostly use the language of homology rather than cohomology. This makes the arguments more visual and so is (within geometric problems like those treated here) more convenient to understand, check and apply the results. Let us recall the main definitions (they are well-known, see [Fe83], pp. 24-30, where also some more details are given). We give equivalent definitions in cohomological language (because all the required properties, if not found in the literature, can either be derived from the cohomology properties or proved directly analogously to them). Cf. [MC06].

Let Q be a compact smooth q-manifold. Let T be a smooth cell-decomposition of Q in the sense of [RS72]. Denote by $H_i(T)$ the corresponding cellular homology groups. Recall that $H_i(Q) := H_i(T)$ is independent of T. Analogously one defines $H_i(Q, \partial Q)$ which we shortly denote by $H_i(Q, \partial)$.

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[10] This ordered set depends on the identification of $H_2(N)$ and $\mathbb{Z}^{b_2(N)}$, but the homotopy type of $C_{BH(f)}$ does not. The homotopy equivalence $C_f \simeq C_{BH(f)}$ is not canonical.

[11] *An alternative direct proof of the Complement Lemma 2.2(b).* Take the map $h_f : C_f \to \mathbb{C}P^\infty$ defined in §4 at the beginning of the proof of the Primitivity Theorem 1.4. Then

$$\pi_3(C_f) \cong \pi_4(\mathbb{C}P^\infty, C_f) \cong H_4(\mathbb{C}P^\infty, C_f) \cong H_4(\mathbb{C}P^\infty)/h_{f,*}H_4(C_f) \cong \mathbb{Z}/d\mathbb{Z}.$$

Here the fourth equality follows because for the dual map $h_f^* : H^4(\mathbb{C}P^\infty) \to H^4(C_f)$ and the generator $a \in H^2(\mathbb{C}P^\infty)$ we have $h_f^*(a \cup a) = h_f^* a \cup h_f^* a = PDA_f \cup PDA_f$. $\square$

The Primitivity Theorem 1.4 and the Complement Lemma 2.2.b imply the following.

• If $H_1(N) = 0$ and $y \in H_2(N)$ is primitive (i.e. there are no integers $d \geq 2$ and elements $x \in H_2(N)$ such that $y = dx$), then $\#BH^{-1}y$ is 0 or 1.

This could be proved analogously to the Primitivity Theorem but using $BH(f) = BH(f')$ and [CS08, Agreement Lemma] instead of $f' = f\#g$. This corollary and the Boéchat-Haefliger Theorem 2.1.a give a proof of the Triviality Theorem 1.1 without reference to the Boéchat-Haefliger Theorem 2.1.b. Cf. [CS08].

• Under the assumptions of the Primitivity Theorem 1.4 the number of isotopy classes of smooth embeddings $f : N \to S^7$ for which $\pi_3(C_f) \cong 0$ equals to the number of primitive elements in $\operatorname{im} BH$.

Let T^* be the dual cell-decomposition. Let $\overline{T}$ be the barycentic subdivision of T. The intersection product $H_i(T)\times H_j(T^*)\to H_{i+j-q}(\overline{T})$ is defined using chain intersections. This gives the intersection product $\cap : H_i(Q)\times H_j(Q)\to H_{i+j-q}(Q)$. Analogously one defines the intersection products $\cap_\partial : H_i(Q,\partial)\times H_j(Q)\to H_{i+j-q}(Q)$ and $\cap_{\partial\partial} : H_i(Q,\partial)\times H_j(Q,\partial)\to H_{i+j-q}(Q,\partial)$.

Denote Poincaré-Lefschetz duality (in any q-manifold Q) by

$$PD : H^i(Q)\to H_{q-i}(Q,\partial)\quad\text{and}\quad PD : H^i(Q,\partial)\to H_{q-i}(Q).$$

Products $\cap' : H^i(Q)\times H_j(Q)\to H_{j-i}(Q)$ is defined e.g. in [Pr07, 8.1]. Analogously one defines $\cap'_\partial : H^i(Q,\partial)\times H_j(Q)\to H_{j-i}(Q)$ and $\cap'_{\partial\partial} : H^i(Q)\times H_j(Q,\partial)\to H_{j-i}(Q,\partial)$. Clearly, $x\cap y = PDx\cap'_\partial y$, $x\cap_{\partial\partial} y = PDx\cup_{\partial\partial} y$ and $x\cap_\partial y = PDx\cap' y$.[12]

In the sequel all the products $\cap,\cap_\partial,\cap_{\partial\partial}$ are denoted simply by $\cap$, the domain of $\cap$ being clear from the context.

Let $p : E\to Q$ be the D^k-bundle associated to a real oriented k-dimensional vector bundle. Let s_* be the zero section. The *'preimage' homomorphism* $s^! : H_i(E)\to H_{i-k}(Q)$ is defined as follows. Take smooth cell-decompositions T_B, T_E of B, E such that s is cellular. Represent a class $x\in H_i(E)$ as a cellular cycle in the dual cell decomposition to T_E. Define $s^![x]$ as the s-preimage of x. Clearly, $s^! = PD\circ s^*\circ PD^{-1}$. *The homology Euler class* of p is defined as $PDe(p) := p_*(s_*[Q]\cap s_*[Q]) = s^!s_*[Q]\in H_{q-k}(Q,\partial)$.[13]

Definition of a weakly unlinked section. Let $\zeta : N_0\to \nu^{-1}N_0$ be a section of the normal bundle $\nu^{-1}N_0\to N_0$. (This exists because $e(\nu)=0$.) Consider the following diagram.

$$H_4(N_0,\partial)\underset{\zeta_*}{\longrightarrow} H_4(\nu^{-1}N_0,\partial)\underset{e}{\leftarrow} H_4(\partial C,\nu^{-1}B^4)\underset{j}{\leftarrow} H_4(\partial C)\underset{i}{\rightarrow} H_4(C).$$

Here j is the isomorphism from the exact sequence of pair, e is the excision isomorphism and i is induced by the inclusion. Section ζ is called *weakly unlinked* if $ij^{-1}e^{-1}\zeta_* = 0$.

Remark 2.4. In the definition of a weakly unlinked section we can replace i by $i' : H_4(\partial C)\to H_4(S^7 - fN_0)$.[14] Indeed, let $\widehat{\nu} : S^7 - \operatorname{Int} C\to N$ be the disk normal bundle.

[12]The intersection products $\cap$ is well-defined, i.e. independent of the triangulation, because $x\cap y = PDx\cap'_\partial y$ and $\cap'_\partial$ is well-defined. (For details of this proof one uses the 'chain-level Poincaré isomorphism' $PD : C_i(T^*)\to C^{q-i}(T)$.) Analogously by passing to cohomology one proves that the other intersection products, the preimage homomorphism $s^!$ and the homology Euler class $PDe(p)$ below are well-defined.

[13]This is clearly equivalent to one of the cohomological definitions $e(p) := s^*s_!PD[Q]\in H^k(E)$. The equality $p_*(s_*[Q]\cap s_*[Q]) = s^!s_*[Q]$ is well-known; here is a proof. Represent $s_*[Q]$ as a cellular cycle q in the dual cell decomposition to T_E. Identify Q with $s(Q)$ by the embedding s. Then both $p_*(s_*[Q]\cap s_*[Q])$ and $s^!s_*[Q]$ are represented by the chain intersection of q with the fundamental chain of $s(Q)$.
Alternative proofs could be obtained either passing to the smooth category and using Thom transversality theorem, or using cohomological definition.

[14]So the definition is equivalent to the following original definition [BH70]. Denote by $|\cdot,\cdot|$ a distance function in N such that B^4 is a ball of radius 2. Define a map

$$\overline{\xi} : N\to S^7 - fN_0\quad\text{by}\quad \overline{\xi}(x) = \begin{cases} \xi(x) & x\in N_0\\ f(x) & |x,N_0|\ge 1\\ |x,N_0|f(x) + (1-|x,N_0|)\xi(x) & |x,N_0|\le 1\end{cases}.$$

Section ζ is called *weakly unlinked* if $\overline{\zeta}_*[N] = 0\in H_4(S^7 - fN_0)$.

The remark follows because inclusion induce isomorphisms $H_4(C)\to H_4(C\cup\widehat\nu^{-1}B^4)\to H_4(S^7-fN_0)$. (The first inclusion is an isomorphism by the exact sequence of pair, the second because it is an inverse to a strong deformation retraction.)

Section Lemma 2.5. *If ζ is a weakly unlinked section, then*
(a) $ej\partial A=\zeta_*[N_0]$.
(b) $BH(f)=PDe(\zeta^\perp)=\zeta^!ej\partial A$, where $\zeta^\perp$ is the oriented S^1-bundle that is the orthogonal complement to ζ in $\nu|_{N_0}$, and for $k\ne0$ we identify $H_k(N)$ with $H_k(N_0,\partial)$ by the composition $H_k(N)\overset{j_N}{\to}H_k(N,B^4)\overset{e_N}{\to}H_k(N_0,\partial)$ of the isomorphism from the exact sequence of pair and the excision isomorphism.

Proof. First we prove (a).[15] Since ζ is weakly unlinked, $j^{-1}e^{-1}\zeta_*[N_0]=\partial x$ for some $x\in H_5(C,\partial)$. By homology Alexander duality $x=kA$ for some integer k. We have $k=1$ because

$$k[N]=\nu_*\partial(kA)=\nu_*j^{-1}e^{-1}\zeta_*[N_0]=\nu|_{N_0,*}\zeta_*[N_0]=[N_0]=[N].$$

Now we prove (b). Observe that the normal bundle ν_ζ of embedding $\zeta:N_0\to\partial C$ is isomorphic to $\zeta^\perp$. Hence their homology Euler classes coincide. Then by (a) we have $\zeta^!ej\partial A=\zeta^!\zeta_*[N_0]=PDe(\nu_\zeta)$. Also,

$$BH(f)\overset{(1)}{=}\nu_*\partial A^2\overset{(2)}{=}\nu|_{N_0,*}ej\partial A^2\overset{(3)}{=}\nu|_{N_0,*}(ej\partial A)^2\overset{(4)}{=}\nu|_{N_0,*}(\zeta_*[N_0])^2\overset{(5)}{=}PDe(\nu_\zeta).$$

Here
- squares denote the intersection squares;
- the first equality is the definition of BH;
- the second equality holds because $\nu_*=\nu|_{N_0,*}ej$;
- the third equality holds by the naturality properties of $\cap$;[16]
- the fourth equality follows by (a);
- the fifth equality is the definition of $PDe(\nu_\zeta)$. $\square$

Compressible embeddings.

Assume that $H_1(N)=0$. A (smooth) embedding $f:N\to\mathbb{R}^7$ is called *PL compressible* if for some embedding $g:S^4\to\mathbb{R}^7$ the embedding $f\#g$ it is isotopic to an embedding $f':N\to\mathbb{R}^7$ such that $f'(N)\subset\mathbb{R}^6$ (this is equivalent to saying that f is PL isotopic to

[15] Cf. [Sk08'], Unlinked Section Lemma (c); in our case a weakly unlinked section need not extend to a section over N. *An alternative proof of (a) using the original definition* is as follows. Take a smooth triangulation of S^7 such that fN_0, fB^4, ∂C, ζN_0 and the union b of segments $f(x)\overline\zeta(x)$, $x\in N$, are subcomplexes. Since ζ is weakly unlinked and $S^7-fN_0\simeq C\cup\widehat\nu^{-1}B^4$, there is a 5-chain a in $C\cup\widehat\nu^{-1}B^4$ such that ∂a is represented by $\overline\zeta N$. (This 5-chain a is in some refinement of the above triangulation, which refinement is used in the rest of this proof.) Recall that 5-chain $a\cap C$ is defined as the sum of simplices of a that are in C. Recall that 5-chain $a\cap(S^7-C)$ is defined as the sum of closures of simplices of a whose interiors are in S^7-C. (These definitions are of course different from the definition of the intersection in homology.) The sum of $a\cap(S^7-C)$ and the 5-chain in $S^7-\mathrm{Int}\,C$ represented by b is a homology between $\partial(a\cap C)$ and the 4-chain in $S^7-\mathrm{Int}\,C$ represented by fN in $S^7-\mathrm{Int}\,C$. Then $\nu_*\partial[a\cap C]=[N]$, so by homology Alexander duality $A=[a\cap C]$. Hence $ej\partial A=[(\nu^{-1}N_0)\cap\partial(a\cap C)]=\zeta_*[N_0]$ (here the intersection of a subset and a chain is defined analogously to the above).

[16] Here is a proof of $\partial A^2=(\partial A)^2$ (analogously one checks that $ej(\partial A)^2=e(j\partial A)^2=(ej\partial A)^2$). Take a triangulation of C. Represent A by a 5-chain a in this triangulation and a 5-chain a' in the dual triangulation. Then $(\partial A)^2$ is represented by $\partial a\cap\partial a'=\partial(a\cap a')$ which represents ∂A^2.

such an embedding f'). Cf. the Effectiveness Theorem 1.2. The study of compressible embeddings is a classical problem in topology of manifolds, see references in [Sk08'].

Remark 2.6. [Vr89] *Let N be a closed connected 4-manifold such that $H_1(N) = 0$.*

(a) An embedding $f : N \to \mathbb{R}^7$ is PL compressible if and only if $BH(f) = 0$ (which holds if and only if $\pi_3(C_f) \cong \mathbb{Z}$).

(b) Two PL compressible embeddings $f : N \to \mathbb{R}^7$ differ only by connected sum with an embedding $S^4 \to \mathbb{R}^7$.

(c) The map $E^6_{PL}(N) \to E^7_{PL}(N)$ induced by the inclusion $\mathbb{R}^6 \to \mathbb{R}^7$ is trivial.

Proof. Part (b) follows from part (a) and the Boéchat-Haefliger Theorem 2.1(b). Part (c) follows from the Boéchat-Haefliger Theorem 2.1(b) and the PL version of (a), which is proved analogously to (a).

Let us prove (a). By the Complement Lemma 2.2(b) $BH(f) = 0$ is equivalent to $\pi_3(C_f) = \mathbb{Z}$.

Clearly, for a PL compressible embedding $f : N \to S^7$ we have $C_f \simeq \Sigma(S^6 - fN)$. Hence $BH(f) = 0$ by the Complement Lemma 2.2(a).

If $BH(f) = 0$, then by the Boéchat-Haefliger Theorem 2.1(a) we have $w_2(N) = 0$ and $\sigma(N) = 0$. Hence there is an embedding $f' : N \to S^6$ [CS79, Ru82, GS99, Remark 9.1.22]. We have $BH(f') = 0$. Hence by the Boéchat-Haefliger Theorem 2.1(b) f is PL compressible. $\square$

3. Attaching invariant and proof of the Effectiveness Theorem 1.2

Recall that *N is a closed connected orientable 4-manifold and $f : N \to \mathbb{R}^7$ is an embedding.* Fix an orientation of N and of $\mathbb{R}^7$. Take a small oriented disk $D^3_f \subset \mathbb{R}^7$ whose intersection with fN consists of exactly one point of sign $+1$ and such that $\partial D^3_f \subset \partial C_f$. *The meridian* S^2_f of f is ∂D^3_f. Identify S^2_f with S^2.

Let G_q be the space of maps $S^{q-1} \to S^{q-1}$ of degree $+1$. The space G_q is identified with a subspace of G_{q+1} via suspension. Let $G = \lim_q G_q$ and $SO = \lim_q SO_q$. The base points are the identity map or its class.

By $*$ we denote the base point of any space. By $[X, Y]$ we denote the set of *based* homotopy classes of maps $X \to Y$ (the choice of base points is clear from the context).

The Haefliger-Levine attaching invariant of knots.

Construction of the attaching invariant $a : E^7(S^4) \to \pi_4(G_3, SO_3)$. Take a smooth embedding $f : S^4 \to S^7$. The space C_f is simply-connected and by Alexander duality the inclusion $S^2_f \to C_f$ induces an isomorphism in homology. Hence this inclusion is a homotopy equivalence. Take a homotopy equivalence h_f homotopy inverse to the inclusion $S^2_f \to C_f$. We may assume that h_f is a retraction onto S^2_f. Since orientations of S^7 and of S^4 are fixed, the homotopy class of h_f depends only on f. Since ν is trivial [Ke59, Ma59], there is a framing $\xi : S^4 \times S^2 \to \partial C_f$ of ν. We may assume that $\xi(* \times S^2) = S^2_f$.

The attaching invariant $a(f, \xi)$ is the homotopy class of the composition

$$S^4 \times S^2 \overset{\xi}{\cong} \partial C_f \subset C_f \overset{h_f}{\simeq} S^2_f.$$

Clearly, $a(f, \xi)$ is independent on isotopy of f. Since $a(f, \xi)|_{* \times S^2} = \mathrm{id}\, S^2$, the map $a(f, \xi)|_{x \times S^2}$ is a homotopy equivalence of degree $+1$ for each x. Hence $a(f, \xi) \in \pi_4(G_3)$.

The choise of a framing ξ is in $\pi_4(SO_3)$. Since the composition $\pi_4(SO_3) \to \pi_4(G_3) \to \pi_4(G_3, SO_3)$ is trivial, it follows that the image $a(f) \in \pi_4(G_3, SO_3)$ of $a(f, \xi)$ does not depend on ξ. This class $a(f)$ is called *the attaching invariant* of f.

Clearly, $a : E^7(S^4) \to \pi_4(G_3, SO_3)$ is a homomorphism. The injectivity of a is proved in §4. The surjectivity of a can be proved analogously, cf. [Fu94]. Cf. Symmetry Remark in §5. We prove the following well-known lemma because we could not find the proof in the literature.

Lemma 3.1. $\pi_5(G, SO) = 0, \quad \pi_4(G, SO) \cong \mathbb{Z}$ and $\pi_4(G_3, SO_3) \cong \pi_6(S^2) \cong \mathbb{Z}_{12}$, cf. [Ha66, the text before Corollary 6.6].

Proof. Let F_q be the space of maps $S^q \to S^q$ of degree $+1$ leaving the north pole fixed. Then $F_q \simeq \Omega_q S^q$, so $\pi_n(F_q) \cong \pi_{n+q}(S^q)$ for $n > 0$. Now from the homotopy exact sequence of the fibration $F_q \to G_{q+1} \to S^q$ we obtain that $\pi_n(G_{q+1}) \cong \pi_n(F_q) \cong \pi_{n+q}(S^q)$ for $q > n + 1$. So $\pi_n(G) = \pi_n^S$.

In order to prove that $\pi_5(G, SO) = 0$ and $\pi_4(G, SO) \cong \mathbb{Z}$ consider the exact sequence of pair

$$\begin{array}{ccccccccccccc}
\pi_5(G) & \to & \pi_5(G, SO) & \to & \pi_4(SO) & \to & \pi_4(G) & \to & \pi_4(G, SO) & \to & \pi_3(SO) & \to & \pi_3(G) \\
\downarrow\cong & & & & \downarrow\cong & & \downarrow\cong & & & & \downarrow\cong & & \downarrow\cong \\
\pi_5^S = 0 & & & & 0 & & \pi_4^S = 0 & & & & \mathbb{Z} & & \pi_3^S \cong \mathbb{Z}_{24}
\end{array}$$

We obtain that $\pi_5(G, SO) = 0$ and $\pi_4(G, SO)$ is isomorphic to a subgroup of $\pi_3(SO) \cong \mathbb{Z}$ having a finite index, i.e. to $\mathbb{Z}$.

In order to prove that $\pi_4(G_3, SO_3) \cong \pi_6(S^2)$ consider the fibration $F_2 \to G_3 \to S^2$ and its subfibration $SO_2 \to SO_3 \to S^2$. We obtain the following diagram

$$\begin{array}{ccccccc}
 & & \pi_3(SO_2) & & \pi_3(SO_3) & & \\
 & & {\scriptstyle 0}\uparrow & & \uparrow & & \\
0 & \to & \pi_4(F_2, SO_2) & \underset{\cong}{\to} & \pi_4(G_3, SO_3) & \to & 0 \\
\uparrow & & {\scriptstyle\cong}\uparrow & & \uparrow & & \uparrow \\
\pi_5(S^2) & \underset{\partial}{\to} & \pi_4(F_2) & \underset{i}{\to} & \pi_4(G_3) & \underset{p}{\to} & \pi_4(S^2) \\
{\scriptstyle =}\uparrow & & {\scriptstyle 0}\uparrow & & \uparrow & & {\scriptstyle =}\uparrow \\
\pi_5(S^2) & \to & \pi_4(SO_2) & \to & \pi_4(SO_3) & \to & \pi_4(S^2)
\end{array}$$

Since $\pi_i(SO_2) = 0$ for $i \geq 2$, $\pi_4(G_3, SO_3) \cong \pi_4(F_2, SO_2) \cong \pi_4(F_2) \cong \pi_6(S^2)$. $\square$

Proof of the Effectiveness Theorem 1.2 and a generalization.

The following lemmas are proved using standard arguments.

A framing $\xi : N_0 \times S^2 \to \partial C_f$ is called *unlinked* if the composition of the section $\xi_1 : N_0 \to \partial C_f$ (formed by first vectors of the framing) with the inclusion $\partial C_f \subset C_f$ is null-homotopic.

Extension Lemma 3.2. *(a) Let $g : S^4 \to \mathbb{R}^7$ and $f : N \to \mathbb{R}^7$ be embeddings such that $f(N) \subset \mathbb{R}^6$. Then there is an unlinked framing $\xi : N_0 \times S^2 \to \partial C_{f\#g}$.*

(b) If $\sigma(N) = 0$, then any framing of $\nu_f|_{N_0}$ extends to that of ν_f.

Proof of (a). Any embedding $N \to \mathbb{R}^6$ has trivial normal bundle. Thus f has a framing $\xi : N_0 \times S^2 \to \partial C_f$ such that the section $\xi_1 : N_0 \to \partial C_f$ formed by third vectors is orthogonal to $\mathbb{R}^6$. Then the composition of ξ_1 with the inclusion $\partial C_f \subset C_f$ is null-homotopic. We may assume that $(f\#g)|_{N_0} = f|_{N_0}$ and $(f\#g)(N - N_0)$ misses the trace of the null-homotopy. Hence $\xi(N_0) \subset \partial C_f \cap \partial C_{f\#g}$ and $\xi : N_0 \times S^2 \to \partial C_{f\#g}$ is an unlinked framing. $\square$

Proof of (b). Given a framing ξ of $\nu_f|_{N_0}$, there is a complete obstruction $O = O(\xi) \in H^4(N, \pi_3(SO_3)) \cong \mathbb{Z}$ to extension of ξ to N. Since the inclusion $SO_3 \subset SO$ induces a multiplication by 2 on π_3, it follows that O equals to twice the obstruction to extension of ξ to a *stable* framing of ν_f. Hence $\pm O = p_1(N) = 3\sigma(N) = 0$ [Ma80, argument before Lemma 1.15]. $\square$

Retraction Lemma 3.3. *Let N be a closed connected orientable 4-manifold, $f : N \to \mathbb{R}^7$ an embedding and $\xi : N_0 \times S^2 \to \partial C_f$ an unlinked framing. Then there is a unique (up to homotopy fixed on S^2_f) retraction $r(\xi) : C_f \to S^2_f$ whose restriction to $\xi(N_0 \times S^2)$ is the projection to $\xi(* \times S^2) = S^2_f$.*

Proof (suggested by a referee). Denote $A := \xi(N_0 \times S^2)$. Since the framing ξ is unlinked, the inclusion $A \to C_f$ extends to a map $A \cup \mathrm{Con}(N_0 \times *) \to C_f$. By the Alexander duality and the Mayer-Vietoris sequence this map induces a homology isomorphism. Hence by the relative Hurewicz Theorem this map is a homotopy equivalence.

Since the projection $A \to S^2$ is null-homotopic on $N_0 \times *$, this projection extends to $A \cup \mathrm{Con}(N_0 \times *)$. This implies the existence in the Retraction Lemma 3.3.

In order to prove the uniqueness in the Retraction Lemma 3.3 consider the Barrat-Puppe exact sequence of sets:

$$[\Sigma A; S^2] \xrightarrow{\rho} [\Sigma(N_0 \times *); S^2] \to [A \cup \mathrm{Con}(N_0 \times *); S^2]_{S^2} \xrightarrow{v} [A; S^2]_{S^2}.$$

Here the index S^2 means that we consider retractions to $S^2 = * \times S^2 \subset A$ up to homotopy fixed on S^2. Since ΣA retracts to $\Sigma(N_0 \times *)$, it follows that ρ is surjective. So by exactness $v^{-1}(*) = 0$. Since v extends to an action of the domain on the range, v is injective. This means that a map $C_f \simeq A \cup \mathrm{Con}(N_0 \times *) \to S^2$ is uniquely defined (up to homotopy fixed on S^2) by its restriction to A. $\square$[17]

[17] The unlinkedness of ξ (or rather the triviality of the induced homology homomorphism ξ_*) is essential in the Retraction Lemma 3.3. Indeed, by the Complement Lemma 2.2(a) $C_f \simeq C_{BH(f)}$, so for $N = \mathbb{C}P^2$ and any embedding $f : \mathbb{C}P^2 \to \mathbb{R}^7$ the space C_f does not retract to S^2. For any framing ξ we have $\xi^* \neq 0$ for $s = 2$ because $BH(f) \mod 2 = w_2(N) \neq 0$.
The homotopy between retractions from Retraction Lemma 3.3 is not assumed to be fixed on $\xi(N_0 \times S^2)$. (The set of retractions $C_f \to S^2$ extending the projection $N_0 \times S^2 \to S^2$ up to homotopy fixed on $N_0 \times S^2$ is in 1–1 correspondence with $H^3(C_f, N_0 \times S^2) \cong H_2(N)$.) We proved that $(C_f, A) \simeq (A \cup \mathrm{Con}(N_0 \times *), A)$; it would be interesting to know if $C_f \simeq \Sigma^2 N_0 \vee S^2$. If $H_1(N) = 0$, then an alternative proof of the Retraction Lemma 3.3 can be obtained because by the Complement Lemma 2.2(a) $C_f \simeq S^2 \vee (\vee_{i=1}^{b_2(N)} S^4)$ (hence retractions $C_f \to S^2_f$ up to homotopy fixed on S^2_f are in 1–1 correspondence with $\mathbb{Z}_2^{b_2(N)}$ or with homomorphisms $H_4(C_f; \mathbb{Z}_2) \to \mathbb{Z}_2$).

Definition of the attaching invariant for an embedding $f : N \to \mathbb{R}^7$ which has an unlinked framing $N_0 \times S^2 \to \partial C_f$. Extend the unlinked framing of f to a framing ξ of f by the Extension Lemma 3.2(b). We may assume that the ξ-image of the base point $* \in N$ is S^2_f. Take the retraction $r = r(\xi|_{N_0})$ given by the Retraction Lemma 3.3. *The attaching invariant* $a(f,\xi)$ is the homotopy class of the composition

$$N \times S^2 \overset{\xi}{\cong} \partial C_f \subset C_f \overset{r(\xi)}{\to} S^2_f.$$

Since $r(\xi)$ is a retraction, $a(f,\xi)|_{*\times S^2} = \operatorname{id} S^2$. Hence $a(f,\xi)|_{x\times S^2}$ is a homotopy equivalence of degree $+1$ for each x. Thus any map representing $a(f,\xi)$ can be identified with base point preserving map $N \to G_3$. Since $a(f,\xi)|_{*\times S^2} = \operatorname{id} S^2$ throughout a homotopy of $r(\xi)$ fixed on S^2_f, we may assume that $a(f,\xi) \in [N, G_3]$.

The choise of a framing ξ is in $[N, SO_3]$. Under a change $\varphi : N \to SO_3$ a map $a : N \to G_3$ changes to the map $a^\varphi : N \to G_3$ defined by $a^\varphi(x) = \varphi(x)a(x)$. Let $\overline{G}_3$ be the the homotopy fiber of $BO_q \to BG_q$. Thus $\pi_i(\overline{G}_3) \cong \pi_i(G_3, SO_3)$ and more generally there is an exact sequence $[N, \Omega BSO_3] \to [N, \Omega BG_3] \to [N, \overline{G}_3]$. Since $\Omega BG_3 \simeq G_3$ and $\Omega BSO_3 \simeq SO_3$, we may assume that $a(f) \in [N, \Omega BG_3]$ and the choise of a framing ξ is in $[N, \Omega BSO_3]$. Hence the image $a(f) \in [N, \overline{G}_3]$ of $a(f,\xi)$ does not depend on ξ. This class $a(f) = a_N(f)$ is called *the attaching invariant* of f.

Additivity Lemma 3.4. *For an embedding $f : N \to \mathbb{R}^7$ which has an unlinked framing $N_0 \times S^2 \to \partial C_f$ and an embedding $g : S^4 \to S^7$ we have $a_N(f\#g) = a_{S^4}(g)\#a_N(f)$, where $\# : \pi_4(\overline{G}_3) \times [N, \overline{G}_3] \to [N, \overline{G}_3]$ is the action given by the map $N \to N/\partial B^4 \simeq N \vee S^4$.*

Proof. Follows because by definition of $a(f,\xi)$ we have $a([f,\xi]\#[g,\zeta]) = a(f,\xi)\#a(g,\zeta)$, where $\# : \pi_4(G_3) \times [N, G_3] \to [N, G_3]$ is the action given by the map $N \to N/\partial B^4 \simeq N \vee S^4$. $\square$

Proof of the Effectiveness Theorem 1.2. Let $f : N \to \mathbb{R}^7$ be an embedding isotopic to $f_0\#g$ for some embedding $g : S^4 \to \mathbb{R}^7$. Take a framing given by the Extension Lemma 3.2(a). Then attaching invariants $a(f,\xi)$ and $a(f)$ are defined.

We have $N/N_0 \cong S^4$. Consider the Barratt-Puppe exact sequence of sets of based homotopy classes:

$$[\Sigma N; \overline{G}_3] \overset{\Sigma R}{\to} [\Sigma N_0; \overline{G}_3] \to \pi_4(\overline{G}_3) \overset{v}{\to} [N; \overline{G}_3] \overset{R}{\to} [N_0; \overline{G}_3].$$

By the Retraction Lemma 3.3 $a(f,\xi)|_{N_0\times S^2}$ is homotopic to the projection onto S^2. Thus the image of $a(f,\xi)$ under the restriction-induced map $[N, G_3] \to [N_0, G_3]$ is the trivial homotopy class. So $a_N(f) \in R^{-1}(*) = \operatorname{im} v$.

Since ΣN retracts to ΣN_0, it follows that ΣR is surjective, so by exactness $v^{-1}(*) = 0$. Since v extends to an action of the domain on the range, v is injective. The map $a_{S^4} : E^7(S^4) \to \pi_4(\overline{G}_3)$ is a monomorphism (this is proved in [Ha66] or at the beginning of §4). So the Theorem follows by the Additivity Lemma 3.4. $\square$[18]

Theorem 3.5. *Let N be a homology n-sphere, $n \geq 3$ and suppose that $\Sigma^\infty : \pi_{n+2}(S^2) \to \pi^S_n$ is not injective. Then for any embedding $f : N \to \mathbb{R}^{n+3}$ there is an embedding $g : S^n \to \mathbb{R}^{n+3}$ such that $f\#g$ is not isotopic to f.*

[18]This proof shows that we can weaken the condition '*N embeds into $\mathbb{R}^6$ and $fN \subset \mathbb{R}^6$*' to '*$fN_0 \subset \mathbb{R}^6$ with trivial normal bundle and $\sigma(N) = 0$*'. Since the triviality of the normal bundle imples that $w_2(N) = 0$, by Remarks in §1 the new assumption still implies that N embeds into S^6.

Proof. Analogously to the construction of the attaching invariant we construct $h_f : C_f \to S^2_f$ and take a framing ξ of ν_f. Now the argument is analogous to the proof of the Effectiveness Theorem 1.2. Since N is a homology sphere, ΣN_0 is contractible. Therefore v is a 1–1 correspondence. Instead of using the injectivity of a_{S^4} we use the exact sequence [Ha66, 4.11, cf. Ha86]

$$(*) \qquad \pi_{n+1}(G,SO) \to E^{n+3}(S^n) \overset{a_{S^n}}{\to} \pi_n(\overline{G}_3) \overset{st}{\to} \pi_n(G,SO),$$

where st is the stabilization map. Recall that st equals to the composition

$$\pi_n(\overline{G}_3) \overset{\cong}{\to} \pi_{n+2}(S^2) \overset{\Sigma^\infty}{\to} \pi_n^S \to \pi_n(G,SO)$$

(Indeed, in the following commutative diagram

$$\begin{tikzcd}
\pi_{n+2}(S^2) \arrow[r, "\cong"'] \arrow[d, "\Sigma^\infty"] & \pi_n(F_2) \arrow[r, "i_*"'] \arrow[d, "st"] & \pi_n(G_3) \arrow[r, "j_*"'] \arrow[d, "st"] & \pi_n(\overline{G}_3) \arrow[d, "st"] \\
\pi_n^S \arrow[r, "\cong"'] & \pi_n(F) \arrow[r, "i_*"'] & \pi_n(G) \arrow[r, "j_*"'] & \pi_n(G,SO)
\end{tikzcd}$$

the upper map j_*i_* is an isomorphism analogously to the proof of $\pi_4(\overline{G}_3) \cong \pi_6(S^2)$.)

Since Σ^∞ is not monomorphic, st is not monomorphic. Thus $\operatorname{im} a_{S^n} \neq 0$. $\square$

The assumption of Theorem 3.5 holds for each $n \leq 19$ except $n \in \{6,7,9,15\}$ [To62, tables]. For $n = 3$ Theorem 3.5 is covered by [Ha72, Ta05]. Analogously to Theorem 3.5 it follows that *for any homology n-sphere N the action of $E^{n+3}(S^n)$ on $E^{n+3}(N)$*

(a) non-trivial, if the stabilization map $\pi_n(\overline{G}_3) \to \pi_{n+1}(G,SO)$ is not injective.

(b) effective, if $\pi_{n+1}(G,SO) = 0$.

4. Proof of the Primitivity Theorem 1.4

Preliminary results.

A map is called *m-connected* if it induces an isomorphism on π_i for $i < m$ and an epimorphism on π_m.

Almost Diffeomorphism Theorem 4.1. *Let C_0 and C_1 be compact simply-connected 7-manifolds and $\varphi : \partial C_0 \to \partial C_1$ a diffeomorphism. For some homotopy 7-sphere Σ there is a diffeomorphism $\varphi : C_0 \to C_1 \# \Sigma$ extending φ if and only if there exist*

- *a fibration $p : B \to BO$ such that $\pi_1(B) = 0$,*
- *a compact 8-submanifold $W \subset S^{18}$ such that $\partial W = M_\varphi := C_0 \cup_\varphi C_1$, and*
- *a lifting $\overline{\nu} : W \to B$ of the classifying map $\nu : W \to BO$ of the normal bundle such that $\overline{\nu}|_{C_0}, \overline{\nu}|_{C_1}$ are 4-connected.*

This is a slight improvement of [Kr99, Theorem 3 and Remark in p. 730] (in which remark for q even and $\pi_1(B) = 0$ we can take $k \geq q-1$). At the end of this section we present the author's write-up of M. Kreck's complete proof of the Almost Diffeomorphism Theorem 4.1.

Denote by $BO\langle m \rangle$ the (unique up to homotopy equivalence) $(m-1)$-connected space for which there exists a fibration $p : BO\langle m \rangle \to BO$ inducing an isomorphism on π_i for

$i \geq m$. (There is a misprint in [Kr99], Definition of k-connected cover on p. 712: $X\langle k\rangle$ should read as $X\langle k+1\rangle$.)

For a fibration $\pi : B \to BO$ denote by $\Omega_q(B)$ the group of bordism classes of liftings $\overline{\mu} : Q \to B$ of the classifying map of stable normal bundle $\mu : Q \to BO$, where Q is a (non-fixed) q-manifold embedded into $\mathbb{R}^{3q}$. Two such liftings $\overline{\mu} : Q \to B$ and $\overline{\mu}' : Q' \to B$ are called *bordant* if there is a $(q+1)$-manifold W and a map $M : W \to B$ such that $\partial W = Q \sqcup Q'$ and $M|_{\partial W} = \overline{\mu} \sqcup \overline{\mu}'$. This should be denoted by $\Omega_q(\pi)$ not $\Omega_q(B)$ but no confusion would appear. (This group is the same as $\Omega_q(B, \pi^* t)$ in the notation of [Ko88].)

Reduction Lemma 4.2. *Embeddings $f, f' : N \to \mathbb{R}^7$ are isotopic if for some orientation-preserving bundle isomorphism $\varphi : \partial C_f \to \partial C_{f'}$ and some embedding $M_\varphi \to S^{16}$ there exist*

- *a space C,*
- *a map $h : M_\varphi \to C$ whose restrictions to C_f and to $C_{f'}$ are 4-connected, and*
- *a lifting $l : M_\varphi \to BO\langle 5\rangle$ of the classifying map $\nu : M_\varphi \to BO$ of the normal bundle such that*

$$[h \times l] = 0 \in \Omega_7(C \times BO\langle 5\rangle)/i_*\theta_7,$$

where θ_7 is the group of irientation-preserving diffeomorphism classes of homotopy 7-spheres.

This situation is explained by the following diagram.

$$\begin{array}{ccc} C \times BO\langle 5\rangle & \xrightarrow[\mathrm{pr}_2]{} & BO\langle 5\rangle \\ \uparrow{\scriptstyle h\times l} & & \downarrow{\scriptstyle p} \\ M_\varphi = C_f \cup_\varphi (-C_{f'}) & \xrightarrow[\mu]{} & BO \end{array}$$

Proof. Denote $B := C \times BO\langle 5\rangle$. Since $[h \times l] = 0$, it follows that there is a homotopy 7-sphere Σ' and a map $s : \Sigma' \to B$ such that $(h\times l)\#s$ is null-bordant. Take a null-bordism $\overline{\nu} : W \to B$ of $(h\times l)\#s$. Since $h|_{C_f}$ and $h|_{C_{f'}}$ are 4-connected and $BO\langle 5\rangle$ is 4-connected, it follows that $\overline{\nu}|_{C_f} = (h\times l)|_{C_f}$ and $\overline{\nu}|_{C_{f'}} = (h\times l)|_{C_{f'}}$ are 4-connected. Therefore by the Almost Diffeomorphism Theorem 4.1 φ extends to an orientation-preserving diffeomorphism $C_f \cong C_{f'}\#\Sigma'\#\Sigma$ for some homotopy sphere Σ. The isomorphism φ also extends to an orientation-preserving diffeomorphism $S^m - \operatorname{Int} C_f \to S^m - \operatorname{Int} C_{f'}$. Therefore $\Sigma'\#\Sigma \cong S^m\#\Sigma'\#\Sigma \cong S^m$. Hence f is isotopic to f' by Lemma 1.3. □[19]

The remaining lemmas of this subsection are essentially known and/or are proved using standard arguments.

Fiber Lemma 4.3. *Let F be the fiber of $p : BO\langle 5\rangle \to BO$. Then*

$$\pi_i(F) = 0 \quad \text{for} \quad i \notin \{1,3\}, \quad \pi_1(F) \cong \mathbb{Z}_2 \quad \text{and} \quad \pi_3(F) \cong \mathbb{Z}.$$

Proof. Recall that $\pi_i(O) \cong \pi_{i+1}(BO) \cong \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}$ according to $i = 0, 1, 2, 3, 4, 5, 6, 7$. From the homotopy exact sequence of the fibration $F \to BO\langle 5\rangle \to BO$ we obtain

[19]This argument shows that in Lemma 1.3 the condition for isotopy could be weakened to ‘there is an orientation-preserving bundle isomorphism $\varphi : \partial C_f \to \partial C_{f'}$ which extends to an orientation-preserving diffeomorphism $C_f \to C_{f'}\#\Sigma$ for some homotopy n-sphere Σ.’

the assertion for $i \ge 4$. From the same exact sequence we obtain $\pi_i(F) \cong \pi_{i+1}(BO) \cong \pi_i(SO) \cong \mathbb{Z}_2, 0, \mathbb{Z}$ for $i = 1, 2, 3$.

Bordism Lemma 4.4. *$\Omega_j(BO\langle 5\rangle) = \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_{24}, 0, 0, \mathbb{Z}_2, 0$ according to $j = 0, 1, 2, 3, 4, 5, 6, 7$.*

Proof. We have $\Omega_j(BO\langle 5\rangle) \cong \Omega_j^{fr}$ for $j \le 6$: for $j \le 4$ because $BO\langle 5\rangle$ is 4-connected and for $j = 5, 6$ because $\pi_j(BO\langle 5\rangle) \cong \pi_j(BO) \cong \pi_{j-1}(O)$.

Each map $Q \to BO\langle 5\rangle$ from a closed 7-manifold Q bordant to a map from homotopy sphere Σ [KM63]. (So $\Omega_7(BO\langle 5\rangle) = i_*\theta_7$, which is already sufficient for the proof of the Primitivity Theorem 1.4.) By [KM63, end of §4] Σ is a boundary of a parallelizable manifold, and $BO\langle 5\rangle$-structure on Σ extends to a $BO\langle 5\rangle$-structure on this manifold. Thus $\Omega_7(BO\langle 5\rangle) = 0$. $\square$

Proof of the injectivity of the attaching invariant $a : E^7(S^4) \to \pi_4(G_3, SO_3)$ defined in §3. Take embeddings $f, f' : S^4 \to \mathbb{R}^7$ such that $a(f) = a(f')$. Then there exist framings ξ and ξ' of ν_f and $\nu_{f'}$ such that $a(f, \xi) = a(f', \xi')$. These framings define an orientation-preserving bundle isomorphism $\varphi : \partial C_f \to \partial C_{f'}$. Identify ∂C_f and $\partial C_{f'}$ by φ. Take any embedding $M_\varphi \to S^{16}$.

Let us set $C = S^2$ in the hypothesis of the Reduction Lemma 4.2. Using obstruction theory and the Fiber Lemma 4.3 we obtain a lifting $l : M_\varphi \to BO\langle 5\rangle$. Recall the definition of homotopy equivalences $h_f : C_f \to S^2$ and $h_{f'} : C_{f'} \to S^2$ from the construction of $a(f)$. Since $a(f, \xi) \simeq a(f', \xi')$, we have $h_f \simeq h_{f'}$ on $\partial C_f = \partial C_{f'}$. Hence by the Borsuk Homotopy Extension Theorem there is a map $h' : C_{f'} \to S^2$ homotopic to $h_{f'}$ and coinciding with h_f on $\partial C_f = \partial C_{f'}$. Set $h = h_f \cup h'$.

We have $\Omega_7(S^2 \times BO\langle 5\rangle) = 0$. (This follows because in the Atiyah-Hirzebruch spectral sequence with $E^2_{i,j} = H_i(S^2, \Omega_j(BO\langle 5\rangle))$ we have by the Bordism Lemma 4.4 $E^2_{i,7-i} = 0$.) Hence $[h \times l] = 0$. Therefore by the Reduction Lemma 4.2 f is isotopic to f'. $\square$

Proof of the Primitivity Theorem 1.4.

Denote $f' = f\#g$. Since the normal bundle of S^4 in S^7 is trivial, there is an orientation-preserving bundle isomorphism $\varphi : \partial C_f \to \partial C_{f'}$ identical over N_0.[20]

Let us set $C = \mathbb{C}P^\infty = K(\mathbb{Z}, 2)$ in the hypothesis of the Reduction Lemma 4.2. Let h_f be any map corresponding to A_f under the bijection $H_5(C_f, \partial C_f) \to [C_f, \mathbb{C}P^\infty]$. Define $h_{f'} : C_{f'} \to \mathbb{C}P^\infty$ analogously. By the Complement Lemma 2.2.b $\pi_3(C_f) = \pi_3(C_{f'}) = 0$. Hence h_f and $h_{f'}$ are 4-connected.

Since $f' = f\#g$, we may assume that $fN_0 = f'N_0$. So by Remark 2.4 a weakly unlinked section for f is a weakly unlinked section for f'. Hence by the Section Lemma 2.5.a (where e and j are isomorphisms) $\partial A_{f'} = \varphi_* \partial A_f$. The restrictions of h_f and $h_{f'}\varphi$ to ∂C_f correspond to ∂A_f and $\varphi_*^{-1}\partial A_{f'}$ under the bijection $H_4(\partial C_f) \to [\partial C_f, \mathbb{C}P^\infty]$. Hence these restrictions are homotopic. Therefore by the Borsuk homotopy extension theorem $h_{f'}$ is homotopic to a map $h' : C_{f'} \to \mathbb{C}P^\infty$ such that $h'\varphi = h_f$ on ∂C_f. Set $h := h_f \cup_\varphi h'$.

Take any embedding $M_\varphi \to S^{16}$. Since $C_f \subset S^7$, it follows that the (stable) normal bundle of C_f is trivial, so $\nu_{M_\varphi}|_{C_f}$ is null-homotopic. Therefore $\nu_{M_\varphi}|_{C_f}$ has a lifting $C_f \to BO\langle 5\rangle$. Obstructions to extending this lifting to M_φ are in $H^{i+1}(C_{f'}, \partial C_{f'}) \cong$

[20] If $H_1(N) = 0$, then there are two orientation-preserving bundle isomorphisms $\nu_f \cong \nu_{f'}$ differing over the top cell of N. We shall prove that *each* of the two is realizable by an isotopy between f and f'.

$H_{6-i}(C_{f'}) \cong H_{4-i}(N)$ with the coefficients $\pi_i(F)$. Since $H_1(N) = 0$, these obstructions are zeroes by the Fiber Lemma 4.3. Thus there is a lifting $l_1 : M_\varphi \to BO\langle 5\rangle$.

By the Reduction Lemma 4.2 it remains to change l_1 to l so that

$$[h \times l] = 0 \in \Omega := \Omega_7(\mathbb{C}P^\infty \times BO\langle 5\rangle)/i_*\theta_7.$$

Consider the Atiyah-Hirzebruch spectral sequence with $E^2_{i,j} = H_i(\mathbb{C}P^\infty, \Omega_j(BO\langle 5\rangle))$. By the Bordism Lemma 4.4 among the groups $E^2_{i,7-i}$ the only non-trivial ones are $E^2_{6,1} \cong \mathbb{Z}_2$ and $E^2_{4,3} \cong \mathbb{Z}_{24}$. By [Te93, Proposition 1] the differential $E^2_{8,0} \to E^2_{6,1}$ is the composition

$$\mathbb{Z} \cong H_8(\mathbb{C}P^\infty) \overset{\rho_2}{\to} H_8(\mathbb{C}P^\infty;\mathbb{Z}_2) \overset{(\mathrm{Sq}^2)^*}{\to} H_6(\mathbb{C}P^\infty;\mathbb{Z}_2) \cong \mathbb{Z}_2.$$

Hence this differential is non-trivial. Let $in : \mathbb{C}P^2 \to \mathbb{C}P^\infty$ be the standard inclusion and $p_5 : BO\langle\infty\rangle \to BO\langle 5\rangle$ the standard map. Since $p_5^* : \Omega_3(BO\langle 5\rangle) \to \Omega_3^{fr}$ is an isomorphism, the map

$$\Omega_3^{fr} \to \Omega \quad \text{defined by} \quad (L : S^3 \to BO\langle\infty\rangle) \mapsto (in \times p_5L : \mathbb{C}P^2 \times S^3 \to \mathbb{C}P^\infty \times BO\langle 5\rangle)$$

is an epimorphism. Hence $[in \times p_5L] = -[h \times l_1]$ for some map $L : S^3 \to BO\langle\infty\rangle$.

Since $h|_{C_f}$ is 4-connected, $(h|_{C_f})_* : H_4(C_f) \to H_4(\mathbb{C}P^\infty)$ is epimorphic. Take $x \in H_4(C_f)$ such that $(h|_{C_f})_*(x) = [\mathbb{C}P^2]$. Below we prove that

(*) *each element of $H_4(C_f)$ is realized by an embedded simply-connected 4-manifold.*

Thus x is represented by a simply-connected 4-submanifold $X \subset C_f$. Let OX be a closed tubular neighborhood of X in M_φ. Denote by D^3 the fiber of the normal D^3-bundle $OX \to X$.

Define a map $l : D^3 \to BO\langle 5\rangle$ so that $l \cup l_1|_{\overline{D}^3}$ would form a map homotopic to p_5L (here $\overline{D}^3$ is D^3 with reversed orientation). Extend l to a map $l : (M_\varphi - \mathrm{Int}\, OX) \cup D^3 \to BO\langle 5\rangle$ as l_1 outside D^3. Obstructions to extending l to M_φ are in $H^{i+1}(OX, D^3 \cup \partial OX; \pi_i(F))$. These obstructions are trivial by the Fiber Lemma 4.3 and Lemma 4.5 below. Hence l extends to a lifting $l : M_\varphi \to BO\langle 5\rangle$. Then $[h \times l] = [h \times l_1] + [in \times p_5L] = 0$. $\square$

Lemma 4.5. *If $OX \to X$ is a D^3-bundle over a closed 4-manifold X such that $H^1(X) = 0$, then $H^{i+1}(OX, D^3 \cup \partial OX) = 0$ for $i = 1, 3$.*

Proof. For $i \neq 1, 3$ consider the exact sequence of triple $(OX, D^3 \cup \partial OX, \partial OX)$:

$$\begin{array}{ccccccc}
H^i(OX,\partial OX) & \to & H^i(D^3\cup\partial OX,\partial OX) & \to & H^{i+1}(OX,D^3\cup\partial OX) & \to & H^{i+1}(OX,\partial OX) \\
\downarrow t & & \downarrow \cong & & & & \downarrow t \\
H^{i-3}(X) & \underset{r}{\to} & H^i(S^3) & & & & H^{i-2}(X)
\end{array}$$

Here t are the Thom isomorphisms. For $i = 1$ the statement is clear. For $i = 3$ the map r is defined by restricting the normal bundle $OX \to X$ to a point, therefore r is an isomorphism. This and $H^1(X) = 0$ imply that $H^4(OX, D^3 \cup \partial OX) = 0$. $\square$

The Alexander Duality Theorem 4.6. *Let $f : N \to S^m$ be an embedding of a closed orientable n-manifold N. The composition $H_{s+n-m+1}(N) \overset{\nu^!}{\to} H_s(\partial C_f) \overset{i}{\to} H_s(C_f)$ of*

the 'preimage' homomorphism (from the Gysin exact sequence) and the inclusion-induced homomorphism is an isomorphism.

This version of Alexander duality is apparently folklore, cf. [BRS76]. It holds because AD is the composition of the preimage (=the Thom), the excision and the boundary isomorphisms:

$$H_{s+n-m+1}(N) \overset{\widehat{\nu}_f^!}{\to} H_{s+1}(S^m - \operatorname{Int} C_f, \partial C_f) \overset{e}{\to} H_{s+1}(S^m, C_f) \overset{\partial}{\to} H_s(C_f).$$

Here $\widehat{\nu}_f : S^m - \operatorname{Int} C_f \to N$ is the D^{m-n}-normal bundle.[21]

Proof of (*). By the Alexander Duality Theorem 4.6 each class in $H_4(C_f)$ can be represented by $\nu^! y$ for some $y \in H_2(N)$. The class y is realizable by an embedded sphere with handles M [Ki89, II, Theorem 1.1]. Hence $\nu^! y = [\nu^{-1} M]$. We make $X := \nu^{-1} M$ simply-connected by embedded surgery as follows. If $\pi_1(X) \neq 0$, then realize a generator of $\pi_1(X)$ by an embedding $k : S^1 \to X$. Since C_f is simply-connected, there is an extension $\overline{k} : D^2 \to C_f$ of k. By general position we may assume that $k \operatorname{Int} D^2 \cap X = \emptyset$ and $\overline{k}$ is an embedding. A framing of kS^1 in X can be extended to a triple of normal linearly independent vector fields on $\overline{k} D^2$ because $\pi_1(V_{5,3}) = 0$. Thus $\overline{k}$ extends to an embedding $\hat{k} : D^2 \times D^3 \to C_f$ such that $\hat{k}(\partial D^2 \times D^3) \subset X$. Let

$$X' := \Big(X - \hat{k}(\partial D^2 \times \operatorname{Int} D^3)\Big) \bigcup_{\hat{k}(\partial D^2 \times \partial D^3)} \hat{k}(D^2 \times \partial D^3),$$

$$\text{so that} \quad [X'] = [X] \in H_4(C_f) \quad \text{and} \quad \pi_1(X') \cong \pi_1(X) / \langle k \rangle .$$

Continuing this procedure we get a simply-connected X. $\square$

Proof of the Almost Diffeomorphism Theorem 4.1.
Consider the following diagram.

$$\begin{tikzcd}
 & K := \ker(H_4(W) \overset{\overline{\nu}_*}{\to} H_4(B)) \arrow[d, "\subset"] & \\
0 \to H_4(C_k) \arrow[r, "i_k"'] & H_4(W) \arrow[r, "j_k"'] \arrow[d, "\overline{\nu}_*"] & V_k := H_4(W, C_k) \to 0 \\
 & H_4(B) &
\end{tikzcd}$$

Consider the following property $(P_{W,\overline{\nu}})$:

[21] In general, none of the homomorphisms $\nu^!, i$ is an isomorphism. This 'homology Alexander duality' is different from [Sk08', the Alexander Duality Lemma]. The isomorphism AD coincides with the 'ordinary' Alexander duality, cf. [BRS76]. Indeed, $AD(x)$ spans the $(s+1)$-cycle $\widehat{\nu}_f^{-1}(x)$, so the intersection (in S^m) of $\widehat{\nu}_f^{-1}(x)$ with any $(n-(s+n-m+1))$-cycle in fN is the same as that of x (in N), and the secondary linking coefficient (in S^m) of $\widehat{\nu}_f^{-1}(x)$ with any torsion $(n-1-(s+n-m+1))$-cycle in fN is the same as that of x (in N).

$\pi_1(B)=0$ and there is a subgroup $U\subset H_4(W)$ such that
- *$U\cap U=0$,*
- *$\overline{\nu}_*U=0$,*
- *$j_k|_U$ is an isomorphism onto an additive direct summand in V_k for $k=0,1$, and*
- *the quotient $j_0U\times V_1/j_1U\to\mathbb{Z}$ of the intersection pairing $\cap:V_0\times V_1\to\mathbb{Z}$ is unimodular.*

Proof of the 'if' part of the Almost Diffeomorphism Theorem 4.1 under the assumption $(P_{W,\overline{\nu}})$. We have $U\subset K$. The form $\cap:K\times K\to\mathbb{Z}$ is even because $x\cap x=\langle w_4(W),x\rangle=\langle\nu^*w_4,x\rangle=\langle w_4,\nu_*x\rangle=\langle w_4,\pi_*\overline{\nu}_*x\rangle=0\mod 2$, where $\pi:BSpin\to BSO$ is the projection and $w_4\in H^4(BSO)$ is the Stiefel Whitney class. So in [Kr99, p.725] we can take $\mu(x):=x\cap x/2$ for $x\in K$ (because 4 is even). We have $Wh(\pi_1(B))=0$ and so an isomorphism is a simple isomorphism.

Hence the hypothesis on U implies that $\theta(W,\overline{\nu})$ is 'elementary omitting the bases' [Kr99, Definition on p. 730 and the second remark on p. 732]. Thus the existence of the required diffeomorphism follows by the h-cobordism theorem and [Kr99, Theorem 3 and second remark on p. 732]. $\square$

Proof that under the assumptions of the Almost Diffeomorphism Theorem 4.1 the property $(P_{W',\overline{\nu}'})$ holds for some compact 8-submanifold $W'\subset S^{18}$ such that $\partial W'=M_\varphi$ and lifting $\overline{\nu}':W'\to B$ of the classifying map $\nu':W'\to BO$ of the normal bundle. Since is C_0 simply-connected and $h|_{C_0}$ is 4-connected, we have that B is simply-connected. After surgery below the middle dimension we may assume that $\overline{\nu}:W\to B$ is 4-connected.

Since both $\overline{\nu}:W\to B$ and its restriction to C_k are 4-connected and the spaces W,B,C_k are simply-connected, it follows that the inclusion induces isomomorphisms $H_i(C_k)\to H_i(W)$ and $H^i(W)\to H^i(C_k)$ for $i<4$. Hence $H_5(W,C_{1-k})\cong H^3(W,C_k)=0$ by Poincaré-Lefschetz duality and the exact sequence of pair (W,C_k).

Since both $\overline{\nu}:W\to B$ and its restriction to C_k are 4-connected, maps $\overline{\nu}_*$ and $\overline{\nu}_*i_k$ on the diagram are surjective. Hence by the Butterfly lemma $j_k|_K$ is surjective. Thus j_k induces an isomorphism $j'_k:V_k\to K/(K\cap\ker j_k)$.

If $j_0x=0$ then $x\cap y=j_0x\cap j_1y=0$ for each $y\in K$. Since $j_1|_K$ is surjective and (by Poincaré-Lefschetz duality) the intersection pairing $\cap:V_0\times V_1\to\mathbb{Z}$ is unimodular, the converse is also true. Hence

$$K\cap\ker j_0=\{x\in K\mid x\cap y=0\text{ for each }y\in K\}=K\cap\ker j_1.$$

Therefore the bilinear form on $K':=K/(K\cap\ker j_0)$ defined by $(a,b)\mapsto a\cap b=j_0a\cap j_1b$ is unimodular. Since $\cap$ is even on K, the form on K' is even. Hence $\sigma(K')$ is divisible by 8. Therefore there is the Kervaire-Milnor framed simply-connected 8-manifold V such that $\sigma(V)=-\sigma(K')$ and ∂V is a homotopy sphere. Stable framing on V and map $\overline{\nu}$ give a map $\overline{\nu}:W\natural V\to B$. We have $\partial(W\natural V)=\partial W\#\partial V$. Change of W to the boundary connected sum $W\natural V$ has the effect of making direct sum of the line $V_0\overset{j_0}{\leftarrow}K\overset{j_1}{\to}V_1$ with $A\overset{id}{\leftarrow}A\overset{id}{\to}A$, where A is the intersection form of V so that $\sigma(A)=-\sigma(K')$. Thus we may assume that $\sigma K'=0$. Now standard argument (for new K') implies that there is a submodule $U'\subset K'$ satisfying to the above four conditions with K,U and j_k replaced with K',U' and j'_k.

(Indeed, there is $a_1\in K'$ such that $a_1\cap a_1=0$. We can choose a_1 to be primitive. By the unimodularity there is $b_1\in K'$ such that $a_1\cap b_1=1$. The restriction of the

intersection form to $\langle a_1, b_1\rangle$ is unimodular. Hence $K/K' = \langle a_1, b_1\rangle \oplus \langle a_1, b_1\rangle^\perp$. We may proceed by the induction to construct a basis $a_1, \dots, a_s, b_1, \dots, b_s$. Set $U' := \langle a_1, \dots, a_s\rangle$. Since elements of the basis are primitive, U' is an additive direct summand. It is also clear that $\cap : U' \times K'/U' \to \mathbb{Z}$ is unimodular.)

Since $H_i(C_1) \to H_i(W)$ are isomorphisms for $i < 4$, we have $H_3(W, C_1) = 0$. Therefore by Poincaré-Lefschetz duality V_0 has trivial torsion. Then $K' \cong V_0$ has trivial torsion. Hence the projection $K \to K'$ has a right inverse ψ. Then $U := \psi U'$ satisfies to the above four conditions. $\square$

Let us make some remarks on higher-dimensional generalizations.

The proof shows that in the Almost Diffeomorphism Theorem 4.1 the dimension 4 can be replaced by $4k$ (and 7 by $8k-1$, 8 by $8k$, 18 by $18k$).

It would be interesting to describe $E^9(N)$ for closed connected 5-manifold N such that $H_1(N) = 0$. This reduces to description of the fibers of the Whitney invariant $E^9(N) \to H_2(N; \mathbb{Z}_2)$ (which is surjective), i.e. the orbits of the action $\mathbb{Z}_2 \cong E^9(S^5) \to E^9(N)$ (here we would need [Kr99, 5.ii]).

It would be interesting to describe $E^8(N)$ for closed connected smooth simply-connected 5-manifolds N. One would need $E^8(S^5) \cong \mathbb{Z}_2$ (not stated in [Ha66, Mi72]).

Proof of $E^8(S^5) \cong \pi_7(S^2) \cong \mathbb{Z}_2$. From the exact sequence $E^9(S^6) \to \theta^3_6 \to \theta_6 \to E^8(S^5) \to \theta^3_5 \to \theta_5$ [Ha66, 4.20], $E^9(S^6) = 0$ [Ha66] and $\theta_5 = \theta_6 = 0$, we obtain that $\theta^3_6 = 0$ and $E^8(S^5) \cong \theta^3_5$. Now the result follows by the following exact sequence

$$\begin{array}{ccccccccccc}
\theta^3_6 & \longrightarrow & \pi_6(G_3, SO_3) & \longrightarrow & P_6 & \longrightarrow & \theta^3_5 & \longrightarrow & \pi_5(G_3, SO_3) & \longrightarrow & P_5 \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & & & & \downarrow\cong & & \downarrow\cong \\
0 & & \pi_8(S^2) \cong \mathbb{Z}_2 & & \mathbb{Z}_2 & & & & \pi_7(S^2) \cong \mathbb{Z}_2 & & 0
\end{array} \quad \square$$

The Whitney invariant $W : E^{6k+4}(S^{2k+1} \times S^{2k+1}) \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is injective because $E^{6k+4}(S^{4k+2}) = 0$ [Mi72]. It would be interesting to know if the element $(1, 1)$ is in its range (the other elements of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ are in the range [Sk06]).

5. Appendix to §3: an alternative proof of the Effectiveness Theorem 1.2

The idea is to construct directly the attaching invariant, which can perhaps be useful for generalizations.

For $X \subset N$ a section $\xi_1 : X \to \partial C_f$ is called *unlinked*, if the composition $X \overset{\xi_1}{\to} \partial C_f \subset C_f$ is null-homotopic. A framing ξ of $\nu|_X$ is *unlinked*, if the section ξ_1 formed by the first vector of ξ is unlinked.

Strong Unlinked Framing Lemma. *If N is simply-connected and f is PL compressible (see end of §2), then*

(a) there is a unique unlinked framing $\xi_f : N \times S^2 \to \partial C_f$ of $\nu = \nu_f$.

(b) each unlinked framing $N_0 \times S^2 \to \partial C_f$ extends to a unique unlinked framing $N \times S^2 \to \partial C_f$.

The compressibility assumption is necessary for (a) but superfluous for (b).

The Strong Unlinked Framing Lemma (a) is proved by first constructing an unlinked framing $N_0 \times S^2 \to \partial C_f$ (the Extension Lemma 3.2(a)), observing that it is unique (see the details below) and then using the Strong Unlinked Framing Lemma (b).

Proof of the uniqueness of framings $N_0 \times S^2 \to \partial C_f$ for simply-connected N. Equivalence classes of framings $N_0 \times S^2 \to \partial C_f$ are in 1–1 correspondence with homotopy classes of maps $N_0 \to SO_3$. Obstructions to homotopy between maps $N \to SO_3$ are in $H^i(N_0, \pi_i(SO_3))$. By duality and since $\pi_2(SO_3) = 0$ and $N_0/\partial N_0 \cong N$, the latter group is zero for each i. Therefore a framing $N_0 \times S^2 \to \partial C_f$ is unique. $\square$

Proof of the Strong Unlinked Framing Lemma (b). Since f is compressible, $\sigma(N) = 0$. Therefore an unlinked framing $N_0 \times S^2 \to \partial C_f$ extends to a framing $N \times S^2 \to \partial C_f$ by the Extension Lemma 3.2(b). Homotopy classes of such extensions ξ and ξ' differ by an element

$$d(\xi, \xi') \in H^4(N; \pi_4(SO_3)) \cong \pi_4(SO_3) \cong \pi_4(S^2) \cong \mathbb{Z}_2.$$

Moreover, for fixed ξ' the correspondence $\xi \mapsto d(\xi, \xi')$ is 1–1. Identify the set F of homotopy classes of such extensions with $\pi_4(S^2)$ by this 1–1 correspondence.

Denote by ξ_1 the section formed by first vectors of ξ and by $l(\xi)$ the homotopy class of the composition $N \xrightarrow{\xi_1} \partial C_f \subset C_f$. Consider the following diagram:

$$\begin{array}{ccccccccc}
 & & & & \pi_4(S^2) & = & F & & \\
 & & & & \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle l} & & \\
[\Sigma N, C_f] & \underset{r}{\to} & [\Sigma N_0, C_f] & \to & \pi_4(C_f) & \underset{v}{\to} & [N, C_f] & \to & [N_0, C_f]
\end{array}$$

Here $j : S^2 = S^2_f \to C_f$ is the inclusion and the bottom line is a segment of the Barratt-Puppe exact sequence of (N, N_0). We do not know that the diagram is commutative but we know that $l(\xi) - l(\xi') = vj_* d(\xi, \xi')$. Hence $l(F)$ is a coset of $\operatorname{im} vj_*$.

Since f is compressible, $BH(f) = 0$. Hence by the Complement Lemma 2.2(a) we have $C_f \simeq C_{BH(f)} \simeq S^2 \vee (\vee_i S^4_i)$. Identify these spaces.

Let us prove the the existence of unlinked framing $N \times S^2 \to \partial C_f$. It suffices to prove that $l(\xi) \subset \operatorname{im} vj_*$ for a framing $\xi \in F$. Since $\xi|_{N_0 \times S^2}$ is unlinked, $l(\xi)|_{N_0}$ is homotopy trivial. Therefore $vx = l(\xi)$ for some $x \in \pi_4(C_f)$. We have $x = j_* y + z$ for some $y \in \pi_4(S^2)$ and $z \in \pi_4(\vee_i S^4_i)$. Since $\xi|_{N_0 \times S^2}$ is unlinked, $\xi_1|_{N_0}$ is unlinked. Since $BH(f) = 0$ and $H_2(N)$ has no torsion, $\xi_1|_{N_0}$ is weakly unlinked (in the sense of §2). So $(vx)_* = \xi_{1,*} : H_4(N) \to H_4(C_f)$ is trivial. Thus $x_* : H_4(S^4) \to H_4(C_f)$ is trivial. Therefore $z = 0$. So $l(\xi) = vx = vj_* y$.

Let us prove the the uniqueness of unlinked framing. Since $\nu|_{N_0}$ is trivial, we have $w_2(N) = 0$. Since N is also simply-connected, by [Mi58] it follows that ΣN retracts to ΣN_0. So r is surjective, hence by exactness v is injective. Since $C_f \simeq S^2 \vee (\vee_i S^4_i)$, we have that j_* is injective. So vj_* is injective. This implies the uniqueness. $\square$

An alternative proof of the Effectiveness Theorem 1.2 (for simply-connected N). By the Strong Unlinked Framing Lemma (a) there is a unique unlinked framing $\xi_f : N \times S^2 \to \partial C_f$. Take a retraction $r_f = r(\xi_f|_{N_0})$ given by the Retraction Lemma 3.3. *The attaching invariant* $a(f)$ is the homotopy class of the composition

$$N \times S^2 \overset{\xi_f}{\cong} \partial C_f \subset C_f \xrightarrow{r_f} S^2_f.$$

Clearly, $a(f)|_{N_0 \times S^2}$ is homotopic to the projection onto the second factor, $a(f)|_{x_0 \times S^2} = \operatorname{id} S^2$ and $a(f)|_{N \times y_0}$ is null-homotopic. Such maps are in canonical 1–1 correspondence

with elements of $\pi_6(S^2)$ by the following Homotopy Lemma. So we have $a(f) \in \pi_6(S^2)$. Now the argument is completed as in the last paragraph of the previous proof (§3). $\square$

Homotopy Lemma. *Let $n \geq 3$ and N be a closed n-manifold such that ΣN retracts to ΣN_0. Denote by X the set of homotopy classes of maps $a : N \times S^2 \to S^2$ for which $a|_{N_0 \times S^2}$ is homotopic to the projection onto the second factor, $a|_{x_0 \times S^2} = \operatorname{id} S^2$ and $a|_{N \times y}$ is null-homotopic. Then there is a canonical 1–1 correspondence $X \to \pi_{n+2}(S^2)$.*

Proof. Maps $N \times S^2 \to S^2$ for which $a|_{x_0 \times S^2} = \operatorname{id} S^2$ can be considered as maps $N \to G_3$. Consider the Barrat-Puppe exact sequence of sets:

$$[\Sigma N; G_3] \xrightarrow{r} [\Sigma N_0; G_3] \to \pi_n(G_3) \xrightarrow{v} [N; G_3] \to [N_0; G_3].$$

Since ΣN retracts to ΣN_0, it follows that r is surjective. So by exactness $v^{-1}(*) = 0$. Since v extends to an action of the domain on the range, v is injective. Therefore v defines a 1–1 correspondence between X and $\ker p$.

Use the notation from the diagram in the proof of $\pi_4(\overline{G}_3) \cong \pi_6(S^2)$ in §3. Since ∂ factors through $\pi_n(SO_2) = 0$, it follows that $\partial = 0$. Hence $\ker p = \operatorname{im} i \cong \pi_n(F_2) \cong \pi_{n+2}(S^2)$. $\square$

Symmetry Remark. *For each embedding $g : S^4 \to \mathbb{R}^7$ the composition of g with the reflection-symmetry $\mathbb{R}^7 \to \mathbb{R}^7$ is isotopic to $-g$.*[22]

Proof. We use the definition of the attaching invariant $a : E^7(S^4) \to \pi_4(\overline{G_3})$ from this section. Recall that a is an isomorphism. Change of orientation of S^7 induces change of orientation of S^2_f. Then under such a change an unlinked section $S^4 \to \partial C_f$ remains unlinked. Since the unlinked framing is defined from the unlinked section using orientations of S^4 and of S^7, it follows that under change of orientation of S^7 the unlinked framing changes orientation at each point. Therefore change of orientation of S^7 carries the attaching invariant $a = a(\psi) : S^4 \times S^2 \to S^2$ to $\sigma_2 \circ a \circ (\operatorname{id} S^4 \times \sigma_2)$, where by $\sigma_m : S^m \to S^m$ we denote the reflection-symmetry. Identify $\pi_4(\overline{G_3})$ with $\pi_6(S^2)$ by Lemma 3.1. Under this identification $\operatorname{id} S^4 \times \sigma_2$ goes to σ_6. Then the required relation follows because

$$\sigma_2 \circ a \circ \sigma_6 = \sigma_2 \circ (-a) = a - [\iota_2, \iota_2] \circ h_0(a) = a - 2\eta \circ h_0(a) = a - 2a = -a \in \pi_6(S^2).$$

Here $h_0 : \pi_6(S^2) \to \pi_6(S^3)$ is the generalized Hopf invariant, which is an isomorphism inverse to the composition with the Hopf map $\eta \in \pi_3(S^2)$ [Po85, Complement to Lecture 6, (10)]. $\square$

Lemma. *For $n \geq 3$ and each embedding $N \to S^{n+3}$ of an n-dimensional homology sphere N there are a unique unlinked section $\xi_{1,f} : N \to \partial C_f$ and a unique unlinked framing $\xi_f : N \times S^2 \to \partial C_f$ (in particular, the normal bundle ν is trivial, cf. [Ke59, Ma59]).*

Proof. The difference elements for sections $N \to \partial C_f$ are in $H^i(N, \pi_i(S^2))$. Hence the sections are in 1-1 correspondence with elements of $\pi_n(S^2)$. Recall that $C_f \simeq S^2$. So there is a unique unlinked section $\xi_{1,f} : N \to \partial C_f$. Since $H^{i+1}(N, \pi_i(S^1)) = 0$ and $H^i(N, \pi_i(S^1)) = 0$, this section can be uniquely extended to an unlinked framing $\xi_f : N \times S^2 \to \partial C_f$. $\square$

[22] By definition, $-g$ is the composition of g with reflection-symmetries of S^4 and of $\mathbb{R}^7$ [Ha66]. So change of the orientation on S^4 alone does not change g.

An alternative proof of Theorem 3.5. Exactly as in the above proof of the Effectiveness Theorem 1.2. We only should replace the Strong Unlinked Lemma by the above Lemma, $\pi_6(S^2)$ by $\pi_{n+2}(S^2)$ and the reference to the last paragraph of the previous proof of the Effectiveness Theorem 1.2 by the reference to the last paragraph of the previous proof of the Theorem 3.5. □

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